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Non-perturbative construction of 2D and 4D supersymmetric Yang-Mills theories with 8 supercharges

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Abstract

In this paper, we consider two-dimensional $\mathcal{N} = (4,4)$ supersymmetric Yang-Mills (SYM) theory and deform it by a mass parameter M with keeping all supercharges. We further add another mass parameter m in a manner to respect two of the eight supercharges and put the deformed theory on a two-dimensional square lattice, on which the two supercharges are exactly preserved. The flat directions of scalar fields are stabilized due to the mass deformations, which gives discrete minima representing fuzzy spheres. We show in the perturbation theory that the lattice continuum limit can be taken without any fine tuning. Around the trivial minimum, this lattice theory serves as a non-perturbative definition of two-dimensional $\mathcal{N} = (4,4)$ SYM theory. We also discuss that the same lattice theory realizes four-dimensional $\mathcal{N} = 2$ $U(k)$ SYM on $\mathbb{R}^2 \times$ (Fuzzy $\mathbb{R}^2$) around the minimum of k-coincident fuzzy spheres.

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1 Introduction

Supersymmetric gauge theories play very important roles in theoretical particle physics. They are not only promising candidates of physics beyond the standard model, which are one of the targets in Large Hadron Collider (LHC) experiments, but also provide crucial insights into non-perturbative aspects of superstring/M theory [1, 2, 3, 4]. Although they are analytically more controllable than non-supersymmetric theories, there have arisen many intriguing features conjectured from various duality arguments which cannot be addressed by current analytic techniques. Therefore it is important to find promising numerical frameworks which enable us to examine them and to obtain new insights into non-perturbative dynamics. However, it is not a straightforward task in lattice field theory because of the notorious difficulties of lattice supersymmetry. So far, for one- and two-dimensional theories [5] and $\mathcal{N} = 1$ pure super Yang-Mills (SYM) theories in three [6] and four dimensions [7], some lattice models are shown to be free from any fine tuning at least perturbatively. (Lattice models for three- and four-dimensional SYM theories are constructed by orbifold or twisting methods preserving some of supersymmetries [5], although they require fine tuning in taking the continuum limit.) In order to overcome this difficulty, one possible direction is to pursue new discretization methods different from conventional lattice.

For one-dimensional theory (matrix quantum mechanics) a powerful "non-lattice" technique [8] is applicable. Maximally supersymmetric matrix quantum mechanics has been studied extensively, and remarkable quantitative agreement with the gauge/gravity duality conjecture has been obtained [9, 10]. (Qualitatively consistent results are obtained also from lattice simulation [11].) [4] For two-dimensional $\mathcal{N} = (2, 2)$ SYM, non-perturbative evidences for the lattice model presented in [14] to require no fine tuning have been given by numerical simulation for the gauge group $G = SU(2)$ in [15] and for $G = SU(N)$ with $N = 2, 3, 4, 5$ in [16]. Furthermore, [17] has shown that the model constructed in [18] is free from the sign problem and gives the same physics as that in [14] after an appropriate treatment of the overall $U(1)$ modes. $\mathcal{N} = (8, 8)$ theory has also been simulated in [19] in order to study the black hole/black string transition. (For other numerical studies in the context of the gauge/gravity duality, see e.g. [20, 21].) Com-

[4] In particular the simulation results are consistent with the existence of the threshold bound state [10], which is an important ingredient of the Matrix theory conjecture [1]. Theories with less supersymmetry have also been studied [12] and the result strongly suggests the threshold bound state does not exist in those cases as expected from the calculation of the Witten index. For $SU(2)$ theory with four supercharges the energy spectrum has been studied in [13] by using the Hamiltonian approach. The simulation results [12] look consistent with the spectrum calculated in [13].

bining the non-lattice or lattice techniques with matrix model/non-commutative space techniques [22, 23], three-dimensional theory can be obtained as a theory on fuzzy sphere [24]. Also, in the planar limit, four-dimensional theory can be obtained using a novel large-N reduction technique [25, 26] inspired by the Eguchi-Kawai equivalence [27]. However, four-dimensional theories of extended supersymmetry at a finite rank of a gauge group were out of reach.[5]

Recently we proposed a new regularization scheme for four-dimensional $\mathcal{N} = 4$ SYM with $G = U(k)$, which is free from fine tuning [29, 30]. It is a hybrid of two-dimensional lattice [31, 14, 32] (see also [33, 18, 34, 35, 36, 37, 38]) and matrix model techniques [24]. We regularized a two-dimensional SYM with plane wave deformation, which has a fuzzy sphere classical solution, and two additional *non-commutative* dimensions are generated by the fuzzy sphere.[6] In four-dimensional $\mathcal{N} = 4$ theory, the commutative limit of non-commutative space is believed to be smooth [41, 42, 43]. Therefore it is expected that we can numerically simulate four-dimensional $\mathcal{N} = 4$ SYM on $\mathbb{R}^4$ using this formulation.

In this paper we provide a non-perturbative formulation of two-dimensional $\mathcal{N} = (4, 4)$ SYM and four-dimensional $\mathcal{N} = 2$ SYM on non-commutative space, which is analogous to the one given in [29]. We first deform the action of two-dimensional $\mathcal{N} = (4, 4)$ SYM by a mass parameter M with keeping *all the supersymmetry*. The gauge group is $U(N)$ or $SU(N)$. As a result of this deformation, flat directions of three scalar fields are stabilized, and not only the trivial configuration but also fuzzy sphere configurations become supersymmetric classical solutions. We further introduce another mass parameter m, which keeps two supercharges, $Q_\pm$, in order to lift up the flat direction of the remaining scalar field. Then the deformed theory is formulated on a two-dimensional square lattice in a manner to preserve $Q_\pm$ exactly. Here we use a prescription developed by one of the authors, F.S. [31, 14]. Thanks to the deformations by M and m, we can solve the problem of the running of the vacuum expectation values of the scalar fields along the flat directions. In this sense, the formulation in this paper can be regarded as a modification of the formulation in [14] to stabilize all the flat directions of scalar fields with keeping $Q_\pm$ supersymmetries. We will give a perturbative argument that the continuum limit of the two-dimensional lattice does not require any fine tuning. Note that the deformation by m does not affect the fuzzy sphere solutions. Namely, they are still solutions preserving $Q_\pm$ in the lattice theory even after introducing m. We next consider the lattice theory with the gauge group $U(N)$ expanded around a k-coincident fuzzy sphere background.

[5] Number of fine tunings in the lattice model has been estimated in [28].

[6] Refs. [39, 40] discuss similar construction of four-dimensional non-commutative spaces from zero-dimensional matrix models.

Taking the lattice continuum limit first, we obtain four-dimensional $\mathcal{N}=2$ SYM on $\mathbb{R}^2\times$(Fuzzy S^2) with the gauge group $U(k)$ deformed by the parameter m. Here the matrix size is given by $N=kn$, the radius of the fuzzy spheres $R=\frac{3}{M}$, the noncommutativity parameter $\Theta=\frac{18}{M^2 n}$, the four-dimensional gauge coupling $g_{4d}^2=2\pi\Theta g_{2d}^2$, and the naturally introduced UV cutoff $\frac{M}{3}(n-1)$. Although the preserved supercharges at this stage are only $Q_\pm$, the supersymmetry breaking is soft by the mass parameter m. Therefore all the supersymmetry is recovered by simply taking the limit of $m\to 0$. Finally, we take a large-N limit sending $M\to 0$ with fixing k and Θ. In this limit, the fuzzy sphere becomes non-commutative Moyal plane $\mathbb{R}^2_\Theta$. Because of the full supersymmetry preserved upon turning off M, we can strongly expect that four-dimensional $\mathcal{N}=2$ SYM on $\mathbb{R}^2\times\mathbb{R}^2_\Theta$ is realized without any fine tuning.

This paper is organized as follows. In the next section, we review continuum $\mathcal{N}=(4,4)$ SYM theory in two dimensions and rewrite the action in the so-called balanced topological field theory (BTFT) form. In the section 3, we add appropriate terms depending on the parameter M to the action, so that all the supercharges are preserved. Furthermore, the mass m is introduced to stabilize all the flat directions. In the section 4, we put the deformed theory on a two-dimensional lattice in a manner to keep two supercharges. In the section 5, we present an intriguing scenario leading to four-dimensional $\mathcal{N}=2$ SYM with a finite-rank gauge group $U(k)$, starting with the two-dimensional lattice formulation given in the section 4. The section 5 is devoted to conclusion and discussion. In the appendix A, we explain how to derive the mass deformation by M that keeps all the supercharges. In the appendix B, the explicit form of the lattice action is presented.

2 Continuum two-dimensional $\mathcal{N}=(4,4)$ supersymmetric Yang-Mills theory

In this section we recast the two-dimensional $\mathcal{N}=(4,4)$ SYM theory on (Euclidean) $\mathbb{R}^2$ with gauge group $G=U(N)$ or $SU(N)$ to a convenient form for our lattice formulation (BTFT form [44]). The action of the theory reads

$$S_{2d}=\frac{2}{g_{2d}^2}\int d^2x\,\mathrm{Tr}\Big[\frac{1}{2}F_{12}^2+\frac{1}{2}\left(\mathcal{D}_\mu X^I\right)^2-\frac{1}{4}[X^I,X^J]^2 \\ +\frac{1}{2}\Psi^T(\mathcal{D}_1+\gamma_2\mathcal{D}_2)\Psi+\frac{i}{2}\Psi^T\gamma_I[X^I,\Psi]\Big], \tag{2.1}$$

where $\mu=1,2$, $I=3,\cdots,6$, $\mathcal{D}_\mu=\partial_\mu+i[A_\mu,\cdot]$, and all the fields are in the adjoint representation of G. They are expanded by a basis of the representation T^a ($a=1,\cdots,\dim(G)$)

as $A_\mu = A_\mu^a T^a, \cdots$. The fermion Ψ is an 8-component spinor which is real, $\Psi^{a*} = \Psi^a$, but not Majorana. The gamma matrices γ_i $(i = 2, \cdots, 6)$ are 8×8 matrices satisfying $\{\gamma_i, \gamma_j\} = -2\delta_{ij}$ and $\gamma_2 \cdots \gamma_6 = -i\mathbf{1}_8$. Their explicit form we use is

$$
\gamma_2 = -i \begin{pmatrix} \sigma_3 & & & \\ & \sigma_3 & & \\ & & \sigma_3 & \\ & & & \sigma_3 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} & -\sigma_2 & & \\ \sigma_2 & & & \\ & & & -\sigma_2 \\ & & \sigma_2 & \end{pmatrix},
$$
$$
\gamma_4 = -i \begin{pmatrix} \sigma_1 & & & \\ & \sigma_1 & & \\ & & \sigma_1 & \\ & & & \sigma_1 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} & & -\sigma_2 & \\ & & & \sigma_2 \\ \sigma_2 & & & \\ & -\sigma_2 & & \end{pmatrix}, \quad \gamma_6 = \begin{pmatrix} & & & -\sigma_2 \\ & & -\sigma_2 & \\ & \sigma_2 & & \\ \sigma_2 & & & \end{pmatrix}, \tag{2.2}
$$

where $\sigma_{1,2,3}$ are Pauli matrices. This theory preserves eight supercharges and the supersymmetry transformation of the fields is given by

$$
\begin{aligned}
\delta' A_1 &= \epsilon^T \Psi, \quad \delta' A_2 = \epsilon^T \gamma_2 \Psi, \quad \delta' X^I = \epsilon^T \gamma_I \Psi, \\
\delta' \Psi &= \left(-F_{12}\gamma_2 - (\mathcal{D}_1 X^I)\gamma_I + (\mathcal{D}_2 X^I)\gamma_{2I} + \frac{i}{2}[X^I, X^J]\gamma_{IJ} \right) \epsilon,
\end{aligned} \tag{2.3}
$$

where ϵ is an 8-component supersymmetry transformation parameter.

As a preparation to construct a lattice theory later, we transcribe the theory in terms of topologically twisted variables. The two-dimensional $\mathcal{N} = (4,4)$ SYM has three global $U(1)$ symmetries: the rotation of (x^1, x^2)-plane $U(1)_E$, the R-symmetry $U(1)_R$, and another rotation $U(1)_V$ whose origin is the chiral rotation in four-dimensional $\mathcal{N} = 2$ SYM from the viewpoint of dimensional reduction. We write the fermion Ψ as

$$
\Psi = \sqrt{2} \left(\xi_R^1, \xi_L^1, \zeta_R^1, \zeta_L^1, \xi_R^2, \xi_L^2, \zeta_R^2, \zeta_L^2 \right)^T, \tag{2.4}
$$

and define the following complex combinations,

$$
\begin{aligned}
&\lambda_R \equiv \xi_R^1 + i\zeta_R^1, \quad \bar\lambda_R \equiv \xi_R^1 - i\zeta_R^1, \quad \lambda_L \equiv \xi_L^1 + i\zeta_L^1, \quad \bar\lambda_L \equiv \xi_L^1 - i\zeta_L^1, \\
&\psi_R \equiv \xi_R^2 + i\zeta_R^2, \quad \bar\psi_R \equiv \xi_R^2 - i\zeta_R^2, \quad \psi_L \equiv \xi_L^2 + i\zeta_L^2, \quad \bar\psi_L \equiv \xi_L^2 - i\zeta_L^2.
\end{aligned} \tag{2.5}
$$

The $U(1)$-charges of the fields are summarized as

Fields	$U(1)_E$	$U(1)_V$	$U(1)_d$	$U(1)_R$
$A_1 \mp iA_2$	± 1	0	± 1	0
$X^3,\ X^4$	0	0	0	0
$X^5 \pm iX^6$	0	0	0	± 2
λ_R	$-\frac{1}{2}$	$\frac{1}{2}$	0	1
λ_L	$\frac{1}{2}$	$\frac{1}{2}$	1	1
$\bar{\lambda}_R$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	-1
$\bar{\lambda}_L$	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1
ψ_R	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	1
ψ_L	$\frac{1}{2}$	$-\frac{1}{2}$	0	1
$\bar{\psi}_R$	$-\frac{1}{2}$	$\frac{1}{2}$	0	-1
$\bar{\psi}_L$	$\frac{1}{2}$	$\frac{1}{2}$	1	-1

where $U(1)_d$ is the diagonal subgroup of $U(1)_E \times U(1)_V$. We rename the fields based on the symmetry $U(1)_d$ as

$$
\begin{aligned}
&B \equiv X^3, \quad C \equiv 2X^4, \quad \phi_\pm \equiv X^5 \pm iX^6, \\
&\lambda_R \equiv \frac{1}{\sqrt{2}}\left(-\chi_+ + \frac{i}{2}\eta_+\right), \quad \lambda_L \equiv \frac{1}{\sqrt{2}}\left(\psi_{+1} - i\psi_{+2}\right), \\
&\bar{\lambda}_R \equiv \frac{1}{\sqrt{2}}\left(\psi_{-1} + i\psi_{-2}\right), \quad \bar{\lambda}_L \equiv \frac{1}{\sqrt{2}}\left(-\chi_- + \frac{i}{2}\eta_-\right), \\
&\psi_R \equiv \frac{1}{\sqrt{2}}\left(\psi_{+1} + i\psi_{+2}\right), \quad \psi_L \equiv \frac{1}{\sqrt{2}}\left(\chi_+ + \frac{i}{2}\eta_+\right), \\
&\bar{\psi}_R \equiv \frac{1}{\sqrt{2}}\left(\chi_- + \frac{i}{2}\eta_-\right), \quad \bar{\psi}_L \equiv \frac{1}{\sqrt{2}}\left(\psi_{-1} - i\psi_{-2}\right).
\end{aligned}
\tag{2.6}
$$

Correspondingly, we define a new expression of the 8-component spinor,

$$
\Psi^{(0)} = \left(\psi_{+1}, \psi_{+2}, \chi_+, \frac{1}{2}\eta_+, \psi_{-1}, \psi_{-2}, \chi_-, \frac{1}{2}\eta_-\right)^T, \tag{2.7}
$$

which is related with Ψ by a unitary transformation,

$$
\Psi = U_8 \Psi^{(0)}, \tag{2.8}
$$

with

$$U_8 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & i & 1 & i & 0 & 0 \\ 1 & -i & 0 & 0 & 0 & 0 & -1 & i \\ 0 & 0 & i & 1 & i & -1 & 0 & 0 \\ -i & -1 & 0 & 0 & 0 & 0 & -i & -1 \\ 1 & i & 0 & 0 & 0 & 0 & 1 & i \\ 0 & 0 & 1 & i & 1 & -i & 0 & 0 \\ -i & 1 & 0 & 0 & 0 & 0 & i & -1 \\ 0 & 0 & -i & 1 & i & 1 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Next we consider the two supercharges, Q'_+ and Q'_-, corresponding to the supersymmetry parameters ϵ_+ and ϵ_-,

$$\epsilon_\pm^T = \epsilon_\pm'^T U_8^{-1}, \tag{2.10}$$

with

$$\epsilon'_+ = \begin{pmatrix} \varepsilon_+ \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \epsilon'_- = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \varepsilon_- \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (\varepsilon_\pm : \text{Grassmann numbers}) \tag{2.11}$$

respectively. The transformation of the fields by $Q'_\pm$ is given by

$$\begin{aligned} &Q'_\pm A_\mu = \psi_{\pm\mu}, \quad Q'_\pm \psi_{\pm\mu} = \pm i\mathcal{D}_\mu \phi_\pm, \quad Q'_\mp \psi_{\pm\mu} = \frac{i}{2}\mathcal{D}_\mu C \mp \widetilde{H}_\mu, \\ &Q'_\pm \widetilde{H}_\mu = [\phi_\pm, \psi_{\mp\mu}] \mp \frac{1}{2}[C, \psi_{\pm\mu}] \mp \frac{i}{2}\mathcal{D}_\mu \eta_\pm, \\ &Q'_\pm B = \chi_\pm, \quad Q'_\pm \chi_\pm = \pm[\phi_\pm, B], \quad Q'_\mp \chi_\pm = \frac{1}{2}[C, B] \mp H, \\ &Q'_\pm H = [\phi_\pm, \chi_\mp] \pm \frac{1}{2}[B, \eta_\pm] \mp \frac{1}{2}[C, \chi_\pm], \\ &Q'_\pm C = \eta_\pm, \quad Q'_\pm \eta_\pm = \pm[\phi_\pm, C], \quad Q'_\mp \eta_\pm = \mp[\phi_+, \phi_-], \\ &Q'_\pm \phi_\pm = 0, \quad Q'_\mp \phi_\pm = \mp \eta_\pm, \end{aligned} \tag{2.12}$$

where H and $\widetilde{H}_\mu$ are auxiliary fields. It is seen that $Q'_\pm$ satisfy

$$Q_+'^2 = (\text{infinitesimal gauge transformation with parameter } \phi_+)\,,$$

$$
\begin{aligned}
Q_-'^2 &= (\text{infinitesimal gauge transformation with parameter } -\phi_-)\,,\\
\left\{Q'_+, Q'_-\right\} &= (\text{infinitesimal gauge transformation with parameter } C)\,.
\end{aligned}
\tag{2.13}
$$

Then we can rewrite the action (2.1) in the so-called BTFT form [44];

$$
S_{2d} = Q'_+ Q'_- \mathcal{F}, \tag{2.14}
$$

with

$$
\mathcal{F} \equiv \frac{1}{g_{2d}^2} \int d^2x \mathrm{Tr}\left[-iB\Phi - \psi_{+\mu}\psi_{-\mu} - \chi_+\chi_- - \frac{1}{4}\eta_+\eta_-\right], \tag{2.15}
$$

where $\Phi = 2F_{12}$. After integrating out the auxiliary fields, the explicit form of the action reads

$$
\begin{aligned}
S_{2d} =& \frac{1}{g_{2d}^2} \int d^2x\, \mathrm{Tr}\left[F_{12}^2 + (\mathcal{D}_\mu B)^2 + \frac{1}{4}(\mathcal{D}_\mu C)^2 + \mathcal{D}_\mu \phi_+ \mathcal{D}_\mu \phi_- \right.\\
&+ \frac{1}{4}[\phi_+, \phi_-]^2 + [B, \phi_+][\phi_-, B] - \frac{1}{4}[B, C]^2 + \frac{1}{4}[C, \phi_+][\phi_-, C]\\
&+ 2i\chi_- \left(\mathcal{D}_1\psi_{+2} - \mathcal{D}_2\psi_{+1}\right) - 2i\chi_+ \left(\mathcal{D}_1\psi_{-2} - \mathcal{D}_2\psi_{-1}\right)\\
&+ 2B\left(\{\psi_{+1}, \psi_{-2}\} - \{\psi_{+2}, \psi_{-1}\}\right)\\
&+ i\eta_+\mathcal{D}_\mu\psi_{-\mu} + i\eta_-\mathcal{D}_\mu\psi_{+\mu} - C\left\{\psi_{+\mu}, \psi_{-\mu}\right\}\\
&- 2\psi_{-\mu}\psi_{-\mu}\phi_+ - 2\psi_{+\mu}\psi_{+\mu}\phi_-\\
&- \chi_-[\phi_+, \chi_-] + \chi_+[\phi_-, \chi_+] + \chi_-[C, \chi_+] - \chi_+[B, \eta_-] - \chi_-[B, \eta_+]\\
&\left.+ \frac{1}{4}\eta_+[\phi_-, \eta_+] - \frac{1}{4}\eta_-[\phi_+, \eta_-] - \frac{1}{4}\eta_+[C, \eta_-]\right].
\end{aligned}
\tag{2.16}
$$

This action is manifestly symmetric under an $SU(2)_R$ subgroup of the R-symmetry group $SU(4)$. The generators of the $SU(2)_R$ are represented as

$$
\begin{aligned}
J_{++} &= \int d^2x \left[\psi_{+\mu}^a(x)\frac{\delta}{\delta\psi_{-\mu}^a(x)} + \chi_+^a(x)\frac{\delta}{\delta\chi_-^a(x)} - \eta_+^a(x)\frac{\delta}{\delta\eta_-^a(x)}\right.\\
&\qquad \left. + 2\phi_+{}^a(x)\frac{\delta}{\delta C^a(x)} - C^a(x)\frac{\delta}{\delta\phi_-{}^a(x)}\right],\\
J_{--} &= \int d^2x \left[\psi_{-\mu}^a(x)\frac{\delta}{\delta\psi_{+\mu}^a(x)} + \chi_-^a(x)\frac{\delta}{\delta\chi_+^a(x)} - \eta_-^a(x)\frac{\delta}{\delta\eta_+^a(x)}\right.\\
&\qquad \left. - 2\phi_-{}^a(x)\frac{\delta}{\delta C^a(x)} + C^a(x)\frac{\delta}{\delta\phi_+{}^a(x)}\right],\\
J_0 &= \int d^2x \left[\psi_{+\mu}^a(x)\frac{\delta}{\delta\psi_{+\mu}^a(x)} - \psi_{-\mu}^a(x)\frac{\delta}{\delta\psi_{\mu}^a(x)} + \chi_+^a(x)\frac{\delta}{\delta\chi_+^a(x)} - \chi_-^a(x)\frac{\delta}{\delta\chi_-^a(x)}\right.
\end{aligned}
$$

$$+\eta_+^a(x)\frac{\delta}{\delta\eta_+^a(x)}-\eta_-^a(x)\frac{\delta}{\delta\eta_-^a(x)}+2\phi_+{}^a(x)\frac{\delta}{\delta\phi_+{}^a(x)}-2\phi_-{}^a(x)\frac{\delta}{\delta\phi_-{}^a(x)}\Bigg], \quad (2.17)$$

which satisfy the $SU(2)$ algebra,

$$[J_0, J_{\pm\pm}] = \pm 2J_{\pm\pm}, \quad [J_{++}, J_{--}] = J_0. \quad (2.18)$$

We see that $(\psi_{+\mu}, \psi_{-\mu})$, (χ_+, χ_-), $(\eta_+, -\eta_-)$ and (Q'_+, Q'_-) transform as doublets and $(\phi_+, C, -\phi_-)$ as a triplet under the $SU(2)_R$ transformation.

3 Mass deformation with keeping 8 supercharges

We next deform the action (2.1) by introducing a mass parameter M;

$$S_{2d,M} = S_{2d} + S_M, \quad (3.1)$$

with

$$\begin{aligned}
S_M =& \frac{2}{g_{2d}^2}\int d^2x \mathrm{Tr}\left[\frac{1}{2}\left(\frac{M}{3}\right)^2 (X^p)^2 - i\frac{M}{6}\Psi^T\gamma_{23}\Psi + i\frac{M}{3}X^3F_{12} + i\frac{M}{3}\epsilon_{pqr}X^pX^qX^r\right] \\
=& \frac{1}{g_{2d}^2}\int d^2x \mathrm{Tr}\left[\frac{M^2}{9}\left(\frac{1}{4}C(x)^2 + \phi_+(x)\phi_-(x)\right) - \frac{M}{2}C(x)\left[\phi_+(x), \phi_-(x)\right]\right. \\
&\left. + i\frac{2M}{3}B(x)F_{12}(x) + \frac{2M}{3}\psi_{+\mu}(x)\psi_{-\mu}(x) + \frac{2M}{3}\chi_+(x)\chi_-(x) - \frac{M}{6}\eta_+(x)\eta_-(x)\right],
\end{aligned} \quad (3.2)$$

where $p, q, r = 4, 5, 6$. This deformation is derived from an eight-supersymmetry analogue [45] of the plane wave matrix model [22]. The derivation is summarized in the appendix A. As discussed there, (3.1) is invariant under the supersymmetry transformation,

$$\delta = \delta' + \delta_M \quad (3.3)$$

with δ' given by (2.3) and

$$\delta_M A_\mu = \delta_M X^I = 0, \quad \delta_M\Psi = -\frac{M}{3}X^p\gamma_p\gamma_{456}\epsilon. \quad (3.4)$$

Namely, the deformed theory (3.1) still preserves eight supercharges.

In order to rewrite (3.1) in the BTFT form, we define the deformed supercharges $Q_\pm$ through the deformed supersymmetry transformation (3.3) and the supersymmetry parameters (2.10). The $Q_\pm$ transformation of the fields is [7]

$$Q_\pm A_\mu = \psi_{\pm\mu}, \quad Q_\pm\psi_{\pm\mu} = \pm i\mathcal{D}_\mu\phi_\pm, \quad Q_\mp\psi_{\pm\mu} = \frac{i}{2}\mathcal{D}_\mu C \mp \widetilde{H}_\mu,$$

[7] The transformation of the auxiliary fields is determined so that relations (3.6) hold.

$$
\begin{aligned}
&Q_\pm \widetilde{H}_\mu = [\phi_\pm, \psi_{\mp\mu}] \mp \frac{1}{2}[C, \psi_{\pm\mu}] \mp \frac{i}{2}\mathcal{D}_\mu \eta_\pm + \frac{M}{3}\psi_{\pm\mu}, \\
&Q_\pm B = \chi_\pm, \quad Q_\pm \chi_\pm = \pm[\phi_\pm, B], \quad Q_\mp \chi_\pm = \frac{1}{2}[C, B] \mp H, \\
&Q_\pm H = [\phi_\pm, \chi_\mp] \pm \frac{1}{2}[B, \eta_\pm] \mp \frac{1}{2}[C, \chi_\pm] + \frac{M}{3}\chi_\pm, \\
&Q_\pm C = \eta_\pm, \quad Q_\pm \eta_\pm = \pm[\phi_\pm, C] + \frac{2M}{3}\phi_\pm, \quad Q_\mp \eta_\pm = \mp[\phi_+, \phi_-] \pm \frac{M}{3}C, \\
&Q_\pm \phi_\pm = 0, \quad Q_\mp \phi_\pm = \mp \eta_\pm.
\end{aligned} \tag{3.5}
$$

We can check that $Q_\pm$ satisfy the nilpotency relations,

$$
\begin{aligned}
Q_+^2 &= (\text{infinitesimal gauge transformation with parameter } \phi_+) + \frac{M}{3}J_{++}, \\
Q_-^2 &= (\text{infinitesimal gauge transformation with parameter } -\phi_-) - \frac{M}{3}J_{--}, \\
\{Q_+, Q_-\} &= (\text{infinitesimal gauge transformation with parameter } C) - \frac{M}{3}J_0.
\end{aligned} \tag{3.6}
$$

Using $Q_\pm$, the action (3.1) is expressed as [8]

$$
S_{2d,M} = \left(Q_+Q_- - \frac{M}{3}\right)\mathcal{F}, \tag{3.7}
$$

where $\mathcal{F}$ is identical with (2.15). Although $S_{2d,M}$ is not precisely Q_+Q_--exact, it is $Q_\pm$-invariant. In fact, since $\mathcal{F}$ is gauge and $SU(2)_R$ invariant,

$$
J_{\pm\pm}\mathcal{F} = J_0\mathcal{F} = 0, \tag{3.8}
$$

and (Q_+, Q_-) is a doublet of $SU(2)_R$,

$$
J_{\pm\pm}Q_\mp = Q_\pm, \quad J_0 Q_\pm = \pm Q_\pm, \tag{3.9}
$$

we see

$$
\begin{aligned}
Q_+ S_{2d,M} &= Q_+^2 Q_- \mathcal{F} - \frac{M}{3}Q_+\mathcal{F} \\
&= \frac{M}{3}J_{++}Q_-\mathcal{F} - \frac{M}{3}Q_+\mathcal{F} = 0, \\
Q_- S_{2d,M} &= \left(\{Q_+, Q_-\}Q_- - Q_+Q_-^2\right)\mathcal{F} - \frac{M}{3}Q_-\mathcal{F} \\
&= -\frac{M}{3}J_0 Q_-\mathcal{F} + \frac{M}{3}Q_+J_{--}\mathcal{F} - \frac{M}{3}Q_-\mathcal{F} = 0.
\end{aligned} \tag{3.10}
$$

[8] This kind of deformation is discussed for various SYM models in [46].

In the next section, we consider a two-dimensional lattice theory corresponding to this theory. If this theory is naively put on a lattice, however, we soon find that it is hard to perform a numerical simulation because of the flat direction along $B(x)$, which causes running of the scalar field. In order to avoid it, we further deform the theory by introducing an additional mass term to $\mathcal{F}$;

$$\mathcal{F} \to \mathcal{F} + \Delta\mathcal{F}, \tag{3.11}$$

with

$$\Delta\mathcal{F} = \frac{1}{g_{2d}^2}\int d^2x \mathrm{Tr}\left(\frac{m}{2}B(x)^2\right), \tag{3.12}$$

where m is a real constant. This deformation clearly preserves the supercharges $Q_\pm$. After these deformations by M and m, the action becomes

$$S_{2d}^{M,m} = \left(Q_+Q_- - \frac{M}{3}\right)(\mathcal{F} + \Delta\mathcal{F}) \tag{3.13}$$

$$\begin{aligned}
=&\frac{1}{g_{2d}^2}\int d^2x\, \mathrm{Tr}\Bigg[F_{12}^2 + (\mathcal{D}_\mu B)^2 + \frac{1}{4}(\mathcal{D}_\mu C)^2 + \mathcal{D}_\mu\phi_+\mathcal{D}_\mu\phi_- \\
&+ \frac{1}{4}[\phi_+,\phi_-]^2 + [B,\phi_+][\phi_-,B] - \frac{1}{4}[B,C]^2 + \frac{1}{4}[C,\phi_+][\phi_-,C] \\
&+ \frac{M^2}{9}\left(\frac{1}{4}C(x)^2 + \phi_+(x)\phi_-(x)\right) - \frac{M}{2}C(x)\left[\phi_+(x),\phi_-(x)\right] \\
&- \frac{m}{2}\left(\frac{M}{3} + \frac{m}{2}\right)B(x)^2 + 2i\left(\frac{M}{3} + \frac{m}{2}\right)B(x)F_{12}(x) \\
&+ 2i\chi_-\left(\mathcal{D}_1\psi_{+2} - \mathcal{D}_2\psi_{+1}\right) - 2i\chi_+\left(\mathcal{D}_1\psi_{-2} - \mathcal{D}_2\psi_{-1}\right) \\
&+ 2B\left(\{\psi_{+1},\psi_{-2}\} - \{\psi_{+2},\psi_{-1}\}\right) \\
&+ i\eta_+\mathcal{D}_\mu\psi_{-\mu} + i\eta_-\mathcal{D}_\mu\psi_{+\mu} - C\left\{\psi_{+\mu},\psi_{-\mu}\right\} \\
&- 2\psi_{-\mu}\psi_{-\mu}\phi_+ - 2\psi_{+\mu}\psi_{+\mu}\phi_- \\
&- \chi_-[\phi_+,\chi_-] + \chi_+[\phi_-,\chi_+] + \chi_-[C,\chi_+] - \chi_+[B,\eta_-] - \chi_-[B,\eta_+] \\
&+ \frac{1}{4}\eta_+[\phi_-,\eta_+] - \frac{1}{4}\eta_-[\phi_+,\eta_-] - \frac{1}{4}\eta_+[C,\eta_-] \\
&+ \frac{2M}{3}\psi_{+\mu}(x)\psi_{-\mu}(x) + \frac{2M}{3}\chi_+(x)\chi_-(x) - \frac{M}{6}\eta_+(x)\eta_-(x)\Bigg].
\end{aligned} \tag{3.14}$$

We see that m must satisfy

$$-\frac{2M}{3} < m < 0, \tag{3.15}$$

in order for $B(x)$ to have a positive mass squared.

Looking at the bosonic part of this action, we see that there are two types of classical solutions; the trivial solution,

$$C(x) = \phi_\pm(x) = B(x) = 0, \tag{3.16}$$

and the fuzzy sphere solution,

$$C(x) = \frac{2M}{3}L_3, \quad \phi_\pm(x) = \frac{M}{3}(L_1 \pm iL_2), \quad B(x) = 0, \tag{3.17}$$

where L_a $(a = 1, 2, 3)$ belong to an N-dimensional (not necessary irreducible) representation of $SU(2)$ generators satisfying $[L_a, L_b] = i\epsilon_{abc}L_c$. Around these solutions, there is no flat direction because of the mass terms. Also, the $Q_\pm$ transformation of $\eta_\pm$,

$$Q_\pm\eta_\pm = \pm[\phi_\pm, C] + \frac{2M}{3}\phi_\pm, \quad Q_\mp\eta_\pm = \mp[\phi_+, \phi_-] \pm \frac{M}{3}C, \tag{3.18}$$

shows that these solutions preserve the $Q_\pm$ supersymmetry. We here emphasize that the shift of m does not affect the fuzzy sphere solution. Furthermore, this solution preserves all the eight supercharges of the continuous theory in the limit of $m \to 0$, as seen from the supersymmetry transformation (A.29) at $m = 0$.

4 Lattice formulation for two-dimensional $\mathcal{N} = (4, 4)$ supersymmetric Yang-Mills theory

In this section we put the deformed theory on a two-dimensional square lattice with lattice spacing a. In this formulation, the supercharges $Q_\pm$ are preserved on the lattice, the gauge field is expressed as a link variable $U_\mu(x) = e^{iaA_\mu(x)} \in G$ as usual lattice gauge theory, and all the other lattice fields are defined on sites and are made dimensionless by multiplying suitable powers of a to the continuum counterparts:

$$\begin{aligned}&(\text{scalars})^{\text{lat}} = a(\text{scalars})^{\text{cont}}, \quad (\text{fermions})^{\text{lat}} = a^{3/2}(\text{fermions})^{\text{cont}},\\&(\text{auxiliary fields})^{\text{lat}} = a^2(\text{auxiliary fields})^{\text{cont}}, \quad Q_\pm^{\text{lat}} = a^{1/2}Q_\pm^{\text{cont}}.\end{aligned} \tag{4.1}$$

Also, dimensionless coupling constants on the lattice are

$$g_0 = ag_{2d}, \quad M_0 = aM, \quad m_0 = am. \tag{4.2}$$

The supersymmetry transformation is realized as

$$Q_\pm U_\mu(x) = i\psi_{\pm\mu}(x)U_\mu(x),$$

$$
\begin{aligned}
Q_\pm \psi_{\pm\mu}(x) =& i\psi_{\pm\mu}(x)\psi_{\pm\mu}(x) \pm i\mathcal{D}_\mu \phi_\pm(x), \\
Q_\pm \psi_{\mp\mu}(x) =& \frac{i}{2}\{\psi_{+\mu}(x), \psi_{-\mu}(x)\} + \frac{i}{2}\mathcal{D}_\mu C(x) \pm \widetilde{H}_\mu(x), \\
Q_\pm \widetilde{H}_\mu(x) =& -\frac{1}{2}\left[\psi_{\mp\mu}(x), \phi_\pm(x) + U_\mu(x)\phi_\pm(x+\hat{\mu})U_\mu(x)^\dagger\right] \\
& \pm \frac{1}{4}\left[\psi_{\pm\mu}(x), C(x) + U_\mu(x)C(x+\hat{\mu})U_\mu(x)^\dagger\right] \\
& \mp \frac{i}{2}\mathcal{D}_\mu \eta_\pm(x) \pm \frac{1}{4}[\psi_{\pm\mu}(x)\psi_{\pm\mu}(x), \psi_{\mp\mu}(x)] \\
& + \frac{i}{2}\left[\psi_{\pm\mu}(x), \widetilde{H}_\mu(x)\right] + \frac{M_0}{3}\psi_{\pm\mu}(x),
\end{aligned}
\tag{4.3}
$$

for the lattice fields $U_\mu(x)$, $\psi_{\pm\mu}(x)$ and $\widetilde{H}_\mu(x)$, and transformation of the other fields is the same as the one in the continuum theory (3.5) with the obvious replacement $M \to M_0$. Here we have used $\mathcal{D}_\mu$ as a covariant forward difference operator,

$$
\mathcal{D}_\mu A(x) \equiv U_\mu(x)A(x+\hat{\mu})U_\mu(x)^\dagger - A(x), \tag{4.4}
$$

for any adjoint field $A(x)$. In order to construct a corresponding lattice action, we take the lattice counterpart of Φ as

$$
\hat{\Phi}(x) = \hat{\Phi}_{U(N)}(x) \equiv \frac{-i\left(U_{12}(x) - U_{21}(x)\right)}{1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2}, \tag{4.5}
$$

for $G = U(N)$ and

$$
\hat{\Phi}(x) = \hat{\Phi}_{SU(N)}(x) \equiv \hat{\Phi}_{U(N)}(x) - \frac{1}{N}\mathrm{Tr}\left(\hat{\Phi}_{U(N)}(x)\right)\mathbf{1}_N, \tag{4.6}
$$

for $G = SU(N)$. Here $U_{\mu\nu}(x) = U_\mu(x)U_\nu(x+\hat{\mu})U_\mu(x+\hat{\nu})^\dagger U_\nu(x)^\dagger$ is a plaquette variable, ϵ is a constant satisfying $0 < \epsilon < 2$ for $G = U(N)$, and $0 < \epsilon < 2\sqrt{2}$ for $N = 2, 3, 4$ and $0 < \epsilon < 2\sqrt{N}\sin(\pi/N)$ for $N \geq 5$ for $G = SU(N)$, and the norm of a matrix is defined by $||A|| = \sqrt{\mathrm{Tr}\,(AA^\dagger)}$ [14].

We then put the two-dimensional theory (3.13) on a lattice using the same form of $\mathcal{F}$ and $\Delta\mathcal{F}$ in (2.15) and (3.12) together with the trivial replacement $\frac{1}{g_{2d}^2}\int d^2x \to \frac{1}{g_0^2}\sum_x$, $M \to M_0$ and $m \to m_0$. The obtained $Q_\pm$-invariant lattice action is

$$
S_{\mathrm{lat}} = \begin{cases} \left(Q_+Q_- - \frac{M_0}{3}\right)\left(\mathcal{F}_{\mathrm{lat}} + \Delta\mathcal{F}_{\mathrm{lat}}\right), & ||1 - U_{12}(x)|| < \epsilon \text{ for } {}^\forall x \\ \infty, & \text{otherwise} \end{cases} \tag{4.7}
$$

with

$$
\mathcal{F}_{\mathrm{lat}} \equiv \frac{1}{g_0^2}\sum_x \mathrm{Tr}\left[-iB(x)\hat{\Phi}(x) - \psi_{+\mu}(x)\psi_{-\mu}(x) - \chi_+(x)\chi_-(x) - \frac{1}{4}\eta_+(x)\eta_-(x)\right],
$$

$$\Delta\mathcal{F}_{\rm lat} \equiv \frac{1}{g_0^2}\sum_x \mathrm{Tr}\left(\frac{m_0}{2}B(x)^2\right). \tag{4.8}$$

The explicit expression of the lattice action is given in the appendix B.

4.1 Absence of flat direction and realization of the physical vacuum

Let us check that the lattice action has the minimum only at the pure gauge configuration $U_{12}(x) = \mathbf{1}_N$ which guarantees that the weak field expansion $U_\mu(x) = 1 + iaA_\mu(x) + \frac{(ia)^2}{2!}A_\mu(x)^2 + \cdots$ is allowed in the continuum limit so that the lattice theory converges to the desired continuum theory at the classical level. After integrating out the auxiliary fields, bosonic part of the action $S_{\rm lat}$ takes the form,

$$S_{\rm lat}^{(B)} = \frac{1}{g_0^2}\sum_x \mathrm{Tr}\left[-\frac{m_0}{2}\left(\frac{M_0}{3}+\frac{m_0}{2}\right)B(x)^2 + i\left(\frac{M_0}{3}+\frac{m_0}{2}\right)B(x)\hat{\Phi}(x)\right] + S_{\rm PDT}, \tag{4.9}$$

where $S_{\rm PDT}$ denotes positive (semi-)definite terms given by (B.5). We will treat the second term, which is purely imaginary, as an operator in the reweighting method, and consider the minimum of the remaining part of $S_{\rm lat}^{(B)}$. If the condition $-\frac{2M_0}{3} < m_0 < 0$, the lattice counterpart of (3.15), is satisfied, the mass terms in (4.9) fix the minimum at

$$B(x) = 0, \tag{4.10}$$

which is independent of $S_{\rm PDT}$. At this minimum, $S_{\rm PDT}$ becomes

$$\begin{aligned} S_{\rm PDT} =& \frac{1}{g_0^2}\sum_x \mathrm{Tr}\left[\sum_\mu (\mathcal{D}_\mu X^p(x))^2 + \left(i[X^p(x), X^q(x)] + \frac{M_0}{3}\epsilon_{pqr}X^r(x)\right)^2\right] \\ &+ \frac{1}{4g_0^2}\sum_x \frac{\mathrm{Tr}\left[-(U_{12}(x)-U_{21}(x))^2\right]}{\left(1-\frac{1}{\epsilon^2}||1-U_{12}(x)||^2\right)^2} \end{aligned} \tag{4.11}$$

with (2.6) for $p, q, r = 4, 5, 6$.

Looking at the first line, we see that the trivial solution (3.16) and the fuzzy sphere solution (3.17) are still classical solutions of the lattice theory by taking into account the replacement $M \to M_0$. In the same manner as in the continuum theory, there is no flat direction around the solutions; we can perform a stable numerical simulation with keeping two supercharges. Note that the fuzzy sphere solution plays a crucial role to discretize four-dimensional $\mathcal{N} = 2$ SYM in the next section.

As discussed in [14], in the last term of (4.11) representing the gauge kinetic term, the admissibility condition singles out the trivial minimum $U_{12}(x) = \mathbf{1}_N$. It shows that the lattice action has a stable physical vacuum and unphysical degeneracies of vacua do not appear.

4.2 Absence of the species doubler

Let us confirm that there is no species doubler in the kinetic terms of this lattice action. Setting $U_\mu(x) = 1$, the kinetic terms for bosons and fermions become

$$\begin{aligned} S_2^{(B)} =& \frac{1}{g_0^2} \sum_x \mathrm{Tr} \left\{ (\Delta_\mu \phi_+(x)) (\Delta_\mu \phi_-(x)) + \frac{1}{4} (\Delta_\mu C(x))^2 + (\Delta_\mu B(x))^2 \right. \\ & \left. + \frac{M_0^2}{9} \left(\phi_+(x)\phi_-(x) + \frac{1}{4} C(x)^2 \right) - \frac{m_0}{2} \left(\frac{M_0}{2} + \frac{m_0}{2} \right) B(x)^2 \right\}, \end{aligned} \tag{4.12}$$

$$S_2^{(F)} = \frac{1}{g_0^2} \sum_x \mathrm{Tr} \left\{ \Psi^{(0)T} G_\mu \frac{1}{2} \left(\Delta_\mu + \Delta_\mu^* \right) \Psi^{(0)} + \Psi^{(0)T} P_\mu \frac{1}{2} \left(\Delta_\mu - \Delta_\mu^* \right) \Psi^{(0)} + \Psi^{(0)T} \mathcal{M} \Psi^{(0)} \right\}, \tag{4.13}$$

respectively. Here Δ_μ and Δ_μ^* are forward and backward difference operators;

$$\Delta_\mu f(x) = f(x + \hat{\mu}) - f(x), \quad \Delta_\mu^* f(x) = f(x) - f(x - \hat{\mu}), \tag{4.14}$$

the matrices G_μ and P_μ are given by

$$\begin{aligned} G_1 = i \begin{pmatrix} & & & \sigma_1 \\ & & -i\sigma_2 & \\ & i\sigma_2 & & \\ \sigma_1 & & & \end{pmatrix}, \quad G_2 = i \begin{pmatrix} & & & -\sigma_3 \\ & & \mathbf{1}_2 & \\ & \mathbf{1}_2 & & \\ -\sigma_3 & & & \end{pmatrix}, \\ P_1 = i \begin{pmatrix} & & & i\sigma_2 \\ & & -\sigma_1 & \\ & \sigma_1 & & \\ i\sigma_2 & & & \end{pmatrix}, \quad P_2 = i \begin{pmatrix} & & & \mathbf{1}_2 \\ & & \sigma_3 & \\ & -\sigma_3 & & \\ -\mathbf{1}_2 & & & \end{pmatrix}, \end{aligned} \tag{4.15}$$

and the mass matrix $\mathcal{M}$ is

$$\mathcal{M} = \begin{pmatrix} & m_d \\ -m_d & \end{pmatrix}, \quad m_d = \mathrm{diag} \left(\frac{M_0}{3}, \frac{M_0}{3}, \frac{M_0}{3} + \frac{m_0}{2}, -\frac{M_0}{3} \right). \tag{4.16}$$

Note that G_μ and P_μ are anti-hermitian matrices and hermitian matrices, respectively, satisfying

$$\{G_\mu, G_\nu\} = -2\delta_{\mu\nu}, \quad \{P_\mu, P_\nu\} = 2\delta_{\mu\nu}, \quad \{G_\mu, P_\nu\} = 0. \tag{4.17}$$

The bosonic part (4.12) takes the form of the standard lattice kinetic terms of bosons; no doubler appears in the bosonic sector. For the fermionic part (4.13), the kernel in the momentum space takes the form,

$$\tilde{\mathcal{D}}_F(p) = \sum_{\mu=1}^{2} \left[iG_\mu \sin(ap_\mu) - 2P_\mu \sin^2 \left(\frac{ap_\mu}{2} \right) \right], \tag{4.18}$$

at $M_0 = m_0 = 0$. Since the mass terms have the same structure as in the continuum, it is sufficient to consider the kinetic terms without the mass terms for our aim. Using (4.17), we can easily see

$$\tilde{\mathcal{D}}_F(p)^2 = \sum_{\mu=1}^{2} 4\sin^2\left(\frac{ap_\mu}{2}\right). \tag{4.19}$$

Since $\tilde{\mathcal{D}}_F(p)$ is hermitian, it shows that only the origin $(p_1, p_2) = (0, 0)$ gives the zero of $\tilde{\mathcal{D}}_F(p)$ in $-\frac{\pi}{a} < p_\mu \le \frac{\pi}{a}$, that is, there is no species doubler in the fermionic sector as well.

4.3 Absence of parameter fine tunings

Next, we discuss in the perturbation theory that the desired quantum continuum theory is obtained without any fine tuning. In the theory near the continuum limit with the auxiliary fields integrated out, let us consider local operators of the type:

$$\mathcal{O}_p(x) = \tilde{M}^{\tilde{m}}\varphi(x)^\alpha \partial^\beta \psi(x)^{2\gamma}, \quad p \equiv \tilde{m} + \alpha + \beta + 3\gamma \tag{4.20}$$

where $\varphi(x)$, $\psi(x)$ and ∂ denote bosonic fields, fermionic fields and derivatives, respectively. $\tilde{M}$ represents M or m. The mass dimension of $\mathcal{O}_p$ is p and $\tilde{m}, \alpha, \beta, \gamma = 0, 1, 2, \cdots$.

From dimensional analysis, radiative corrections from ultraviolet (UV) region of loop momenta to $\mathcal{O}_p$ have the form,

$$\left(\frac{1}{g_{2d}^2}c_0\, a^{p-4} + c_1\, a^{p-2} + g_{2d}^2 c_2\, a^p + \cdots\right)\int d^2x\mathcal{O}_p(x), \tag{4.21}$$

up to possible powers of $\ln(a\tilde{M})$. c_0, c_1, c_2 are dimensionless numerical constants. The first, second and third terms in the parenthesis are contributions from tree, 1-loop and 2-loop effects, respectively. The "$\cdots$" is an effect from higher loops, which are irrelevant for the analysis.

Since relevant or marginal operators generated by loop effects possibly appear from nonpositive powers of a in the second and third terms in (4.21), we should look at operators with $p = 0, 1, 2$. They are φ, $\tilde{M}\varphi$ and φ^2 except for non-dynamical operators like 1, $\tilde{M}$, $\tilde{M}^2$ and $\partial\varphi$. For $G = U(N)$, although only the candidates for φ is $\mathrm{Tr}\, B$ from gauge and $SU(2)_R$ symmetries, it is not invariant under $Q_\pm$ supersymmetries; it is forbidden to appear. Similarly, $\tilde{M}\varphi$ and φ^2 are not allowed to be radiatively generated. For $G = SU(N)$, we may consider φ^2 alone, whose candidates are not generated by the symmetries.

Therefore, in the perturbative argument, we can conclude that any relevant or marginal operators except non-dynamical operators do not appear radiatively, meaning that no fine tuning is required to take the continuum limit. In particular, if we consider the lattice

theory around the trivial minimum $C = \phi_\pm = 0$, the mass-deformed two-dimensional $\mathcal{N} = (4,4)$ SYM is obtained without any fine-tuning. Also, after taking the limit of $m \to 0$, we can safely take the limit of $M \to 0$ to reach the undeformed theory because of the exactly preserved eight supersymmetries. Thus we can use this lattice theory as a non-perturbative definition of two-dimensional $\mathcal{N} = (4,4)$ SYM theory.

5 Four-dimensional $\mathcal{N} = 2$ supersymmetric Yang-Mills theory in the non-commutative space

In this section, we discuss a scenario to obtain four-dimensional $\mathcal{N} = 2$ SYM on $\mathbb{R}^2 \times$ (Fuzzy $\mathbb{R}^2$) from the lattice formulation given in the previous section.

Let us consider the lattice theory for $G = U(N)$ around the minimum of k-coincident fuzzy sphere solution (3.17) with M replaced by M_0,

$$L_a = L_a^{(n)} \otimes \mathbf{1}_k \quad (a = 1,2,3) \quad \text{and} \quad N = nk. \tag{5.1}$$

$L_a^{(n)}$ are generators of an $n(= 2j+1)$-dimensional irreducible representation of $su(2)$ corresponding to spin j.

First, we take the continuum limit of the two-dimensional lattice theory. Then, we obtain four-dimensional $\mathcal{N} = 2$ $U(k)$ SYM on $\mathbb{R}^2 \times$ (Fuzzy S^2) deformed by the mass parameter m. The fuzzy S^2 has the radius $R = \frac{3}{M}$ and its noncommutativity is characterized by the parameter $\Theta = \frac{18}{M^2 n}$. The UV cutoff Λ is naturally introduced by the size of the matrix; $\Lambda = \frac{M}{3} \cdot 2j$. Although these properties of the fuzzy S^2 can be seen by doing a similar calculation as presented in Refs. [47, 24, 25], let us give a brief argument. Momentum modes of a field, say B, on two dimensions are expanded further by fuzzy spherical harmonics:

$$\tilde{B}(q) = \sum_{J=0}^{2j} \sum_{m=-J}^{J} \hat{Y}_{J\,m}^{(jj)} \otimes b_{J\,m}(q), \tag{5.2}$$

corresponding to the expression (5.1). The fuzzy spherical harmonic $\hat{Y}_{J\,m}^{(jj)}$ is an $n \times n$ matrix whose elements are given by Clebsch-Gordon coefficients as [25]

$$\hat{Y}_{J\,m}^{(jj)} = \sqrt{n} \sum_{r,r'=-j}^{j} (-1)^{-j+r'} C_{j\,r\,j\,-r'}^{J\,m} \, |j\,r\rangle\langle j\,r'| \tag{5.3}$$

with an orthonormal basis $|j\,r\rangle$ representing $L_a^{(n)}$ in the standard way:

$$\left(L_1^{(n)} \pm i L_2^{(n)}\right) |j\,r\rangle \;\; = \;\; \sqrt{(j \mp r)(j \pm r + 1)}\, |j\,r \pm 1\rangle,$$

$$L_3^{(n)}\,|j\,r\rangle \;\;=\;\; r\,|j\,r\rangle, \tag{5.4}$$

and the modes $b_{J\,m}(q)$ are $k\times k$ matrices. It is seen that the fuzzy spherical harmonics are eigen-modes of the Laplacian on the fuzzy S^2:

$$\sum_{a=1}^{3}\left(\frac{M}{3}\right)^2[L_a^{(n)},[L_a^{(n)},\hat{Y}_{J\,m}^{(jj)}]]=\left(\frac{M}{3}\right)^2 J(J+1)\hat{Y}_{J\,m}^{(jj)}, \tag{5.5}$$

giving the rotational energy with the angular momentum J on the sphere of the radius $R=\frac{3}{M}$. The UV cutoff $\Lambda=\frac{M}{3}\cdot 2j$ can be read off from the upper limit of the sum of J in the expansion (5.2). The fuzzy S^2 is a two-dimensional non-commutative space, which is analogous to the phase space of some one-dimensional quantum system, and the noncommutativity Θ to the Planck constant $\hbar$. The quantum phase space is divided into small cells of the size $2\pi\hbar$, whose number is equal to the dimension of the Hilbert space. Correspondingly, the area of the S^2 is divided into n cells of the size $2\pi\Theta$:

$$4\pi R^2=n\cdot 2\pi\Theta, \tag{5.6}$$

leading to the value $\Theta=\frac{18}{M^2 n}$.

As stressed in the previous section, the supersymmetry is softly broken from eight to two because of the mass parameter m at this stage. The eight supercharges are recovered by taking the limit of $m\to 0$ with fixing M.

Next we take the limit of $n\to\infty$ with fixing Θ and k. In this limit, M and Λ become

$$M\propto n^{-1/2}\to 0,\quad \Lambda\propto n^{1/2}\to\infty, \tag{5.7}$$

and the fuzzy S^2 is decompactified to the non-commutative Moyal plane $\mathbb{R}^2_\Theta$. Since the fuzzy S^2 solution preserves eight supercharges after taking $m\to 0$, it is strongly expected that the theory becomes $\mathcal{N}=2$ $U(k)$ SYM on $\mathbb{R}^2\times\mathbb{R}^2_\Theta$ after taking the above limit. The gauge coupling constant of the four-dimensional theory is given in the form

$$g_{4d}^2=2\pi\Theta g_{2d}^2. \tag{5.8}$$

After taking this limit, the expansion (5.2) by the fuzzy spherical harmonics can be essentially transcribed to the one by plane waves on $\mathbb{R}^2_\Theta$:

$$\tilde{B}(q)=\int\frac{d^2\tilde{q}}{(2\pi)^2}\,e^{i\tilde{q}\cdot\hat{x}}\otimes\tilde{b}(\mathbf{q}), \tag{5.9}$$

where q and $\tilde{q}$ are two-momenta on $\mathbb{R}^2$ and $\mathbb{R}^2_\Theta$ respectively, the position operator $\hat{x}=(\hat{x}_1,\hat{x}_2)$ on $\mathbb{R}^2_\Theta$ satisfies $[\hat{x}_1,\hat{x}_2]=i\Theta$, and $\mathbf{q}\equiv(q,\tilde{q})$ represents a four-momentum. The

modes $\tilde{b}(\mathbf{q})$ in the four-dimensional space are $k \times k$ matrices. It is easy to calculate the inner product between plane waves on $\mathbb{R}^2_\Theta$:

$$\mathrm{Tr}\,\left(e^{i\tilde{p}\cdot\hat{x}}e^{i\tilde{q}\cdot\hat{x}}\right) = \frac{2\pi}{\Theta}\,\delta^2(\tilde{p}+\tilde{q}), \tag{5.10}$$

which leads to the Θ-dependence of the relation (5.8).

Although more investigation is needed to clarify whether the $\Theta \to 0$ limit of the theory is continuously connected to the commutative four-dimensional $\mathcal{N} = 2$ SYM on $\mathbb{R}^4$ or not[9], to the best of our knowledge this formulation gives the first non-perturbative formulation free from fine tuning for four-dimensional SYM with eight supercharges.

6 Conclusion and discussion

In this paper, we deformed two-dimensional $\mathcal{N} = (4,4)$ SYM theory with the gauge group $U(N)$ or $SU(N)$ by a mass parameter M with preserving all supercharges and expressed the deformed action in BTFT form. We further deformed it by introducing an additional mass parameter m in a manner to keep two supercharges, $Q_\pm$, in order to lift up all the flat directions of the scalar fields. We then put the deformed theory on a two-dimensional lattice with preserving $Q_\pm$ exactly. The problem of the running of the vacuum expectation values of the scalar fields is avoided thanks to the deformation by M and m. We also gave a perturbative argument that any fine tuning is not needed in taking the lattice continuum limit. Thus this lattice theory around the trivial minimum can be regarded as a non-perturbative definition of two-dimensional $\mathcal{N} = (4,4)$ SYM theory. To perform actual numerical simulation, it should be checked if the imaginary term of the bosonic action (4.9) is managed by the reweighting method, which might cause bosonic sign problem independent of the fermionic one. We next considered the lattice theory for the gauge group $U(N)$ around a k-coincident fuzzy sphere solution with $N = nk$. The radius of the fuzzy S^2 is $\frac{3}{M}$ and the noncommutativity of the fuzzy sphere is characterized by the parameter $\Theta = \frac{18}{M^2 n}$. By taking the lattice continuum limit, we obtained four-dimensional $\mathcal{N} = 2$ $U(k)$ SYM on $\mathbb{R}^2 \times$ (Fuzzy S^2) deformed by the mass parameter m. It was discussed that, by taking the limit of $m \to 0$ followed by the limit of $M \to 0$ with fixing k and Θ, four-dimensional $\mathcal{N} = 2$ SYM on $\mathbb{R}^2 \times \mathbb{R}^2_\Theta$ is realized without any fine tuning.

[9] It is naively expected that the $\Theta \to 0$ limit would not be continuously connected to the commutative theory because of the ultraviolet/infrared (UV/IR) mixing [48]. There is a discussion, however, that non-commutative four-dimensional $\mathcal{N} = 2$ $U(k)$ SYM may flow to the ordinary commutative theory in the infrared [49].

In contrast to four-dimensional $\mathcal{N}=4$ SYM, the commutative limit $\Theta \to 0$ of four-dimensional $\mathcal{N}=2$ non-commutative SYM is expected not to be continuously connected to the usual commutative $\mathcal{N}=2$ SYM because of UV/IR mixing. Even if such expectation is true and our scenario does not lead to $\mathcal{N}=2$ SYM on the usual $\mathbb{R}^4$, notice that non-commutative gauge theory itself is an important subject of research in order to clarify non-perturbative aspects of gauge theories. For example, when we consider instantons of gauge theories, noncommutativity plays a crucial role to resolve the small instanton singularity. In our formulation, we can analyze the dynamical aspect of quite wide class of observables of four-dimensional $\mathcal{N}=2$ non-commutative SYM numerically, which will give a strong instrument to reveal the non-perturbative structure of supersymmetric gauge theories.

The deformed two-dimensional theory itself is also interesting on its own. In particular, since one can introduce mass terms for all scalars keeping two supersymmetries, and hence flat directions (along which all scalars commute each other) are all lifted, one can perform stable Monte-Carlo simulation, if reweighting for the imaginary term in (4.9) works. (Simulations so far utilized supersymmetry breaking mass terms, which make the conclusion more or less obscure.) Moreover, the deformation terms consist of mass terms and a Myers term, which are quite similar to the so-called Ω-deformation [50, 51] which is originally introduced in order to regularize the instanton moduli space of four-dimensional $\mathcal{N}=2$ SYM theory[10]. In the case of the Ω-deformation, the integration over the instanton moduli space is localized to discrete points, which makes it possible to evaluate the instanton partition function analytically using localization formula in equivariant cohomology. On the other hand, the deformation introduced in this paper lifts flat directions of the scalar fields, which makes it possible to carry out stable Monte-Carlo simulation. It is interesting that a mathematically sophisticated technique like equivariant cohomology somehow relates to a technique developed for numerical simulation in this paper. It may be a sign that such a mathematical method would give a systematic method to construct a non-perturbative definition of supersymmetric gauge theories in the future.

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[10] In fact, starting with one-dimensional theory that is obtained by the dimensional reduction of four-dimensional $\mathcal{N}=2$ SYM in the Ω-background, we can construct a regularized three-dimensional theory on $\mathbb{R} \times$ (Fuzzy S^2) with keeping at least a part of the supersymmetry [52].

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A Plane wave deformed two-dimensional $\mathcal{N} = (4,4)$ supersymmetric Yang-Mills theory

In this appendix we explain how to construct the plane wave deformed two-dimensional $\mathcal{N} = (4,4)$ supersymmetric Yang-Mills theory.

A.1 BMN type matrix model with 8 supercharges

Let us start with an eight-supersymmetry analogue [45] of the plane wave matrix model [22],

$$\begin{aligned} S &= R\int dt\,\mathrm{Tr}\left[\frac{1}{2R^2}(D_tX^i)^2 + \frac{i}{R}\Psi^T D_t\Psi + \Psi^T\Gamma^i[X^i,\Psi] + \frac{1}{4}[X^i,X^j]^2 \right. \\ &\left. \quad -\frac{1}{2}\left(\frac{\mu}{3R}\right)^2(X^a)^2 - \frac{1}{2}\left(\frac{\mu}{6R}\right)^2(X^{a'})^2 - i\frac{\mu}{4R}\Psi^T\Gamma^{456}\Psi - i\frac{\mu}{3R}\epsilon_{abc}X^aX^bX^c\right], \end{aligned} \tag{A.1}$$

where $D_t = \partial_t + i[A_t,\ \cdot\ ]$, $i = 2,3,4,5,6$, $a = 4,5,6$ and $a' = 2,3$. Γ^i are 8×8 real symmetric matrices corresponding to $-i\gamma_i$, which satisfy

$$\{\Gamma^i,\Gamma^j\} = 2\delta^{ij}\mathbf{1}_8, \tag{A.2}$$

$$\Gamma^{23456} = \Gamma^2\cdots\Gamma^6 = -1. \tag{A.3}$$

From this model we construct a two-dimensional theory following [53], by using Taylor's T-duality. In order to lift the theory to two dimensions, we redefine the fields by a rotation on (X^2,X^3) plane with the angle αt as

$$\begin{pmatrix} X^2 \\ X^3 \end{pmatrix} = U_\alpha(t)\begin{pmatrix} \hat{X}^2 \\ \hat{X}^3 \end{pmatrix}, \qquad \Psi = e^{\frac{1}{2}\Gamma^{23}\alpha t}\hat{\Psi} \tag{A.4}$$

with

$$U_\alpha(t) \equiv \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix}. \tag{A.5}$$

For the other variables, the hatted variables are the same as the unhatted ones. For example, $D_t = \partial_t + i[A_t, \cdot] = \partial_t + i[\hat{A}_t, \cdot]$. Since Γ^{23} is real anti-symmetric, Ψ^T transforms as $\Psi^T = \hat{\Psi}^T e^{-\frac{1}{2}\Gamma^{23}\alpha t}$. The action in terms of the redefined fields is

$$
\begin{aligned}
S \;=\; & R\int dt\,\mathrm{Tr}\,\Bigg[\frac{1}{2R^2}(D_t\hat{X}^i)^2 + \frac{i}{R}\hat{\Psi}^T D_t\hat{\Psi} + \hat{\Psi}^T\Gamma^i[\hat{X}^i,\hat{\Psi}] + \frac{1}{4}[\hat{X}^i,\hat{X}^j]^2 \\
& -\frac{1}{2}\left\{\left(\frac{\mu}{6R}\right)^2 - \frac{\alpha^2}{R^2}\right\}(\hat{X}^{a'})^2 - i\frac{\mu}{4R}\hat{\Psi}^T\left(\Gamma^{456} - \frac{2\alpha}{\mu}\Gamma^{23}\right)\hat{\Psi} \\
& +\frac{2\alpha}{R^2}\hat{X}^3 D_t\hat{X}^2 - \frac{1}{2}\left(\frac{\mu}{3R}\right)^2(\hat{X}^a)^2 - i\frac{\mu}{3R}\epsilon_{abc}\hat{X}^a\hat{X}^b\hat{X}^c\Bigg] \\
& -\frac{\alpha}{R}\int dt\,\partial_t\mathrm{Tr}\,(\hat{X}^2\hat{X}^3). \qquad \text{(A.6)}
\end{aligned}
$$

If we set $\alpha = \pm\frac{\mu}{6}$ and discard the surface term, the mass terms of $\hat{X}^{a'}$ vanish, and $\hat{X}^2$ appears only in the adjoint form. Then, the Taylor's T-duality may be performed with respect to $\hat{X}^2$. As we will see shortly, compatibility of supersymmetry transformation and T-duality singles out $\alpha = -\frac{\mu}{6}$.

A.1.1 Supersymmetry transformation

Supersymmetry transformation of this matrix quantum mechanics is given by

$$
\begin{aligned}
\delta X^i &= i\Psi^T\Gamma^i\epsilon(t), \\
\delta A_t &= Ri\Psi^T\epsilon(t), \\
\delta\Psi &= \left\{\frac{1}{2R}(D_tX^i)\Gamma^i + \frac{i}{4}[X^i,X^j]\Gamma^{ij}\right. \\
& \qquad \left. +\frac{\mu}{6R}X^a\Gamma^a\Gamma^{456} - \frac{\mu}{12R}X^{a'}\Gamma^{a'}\Gamma^{456}\right\}\epsilon(t) \qquad \text{(A.7)}
\end{aligned}
$$

with

$$\epsilon(t) = e^{-\frac{\mu}{12}\Gamma^{456}t}\epsilon_0, \qquad \text{(A.8)}$$

where ϵ_0 is an 8-component constant spinor. This is called "dynamical supersymmetry" which is simply referred as supersymmetry in the text. For the case of $G = U(N)$, it is also invariant under "kinematical supersymmetry",

$$\tilde{\delta}\Psi = \eta(t)\mathbf{1}_N \qquad \text{(A.9)}$$

with

$$\eta(t) = e^{\frac{\mu}{4}\Gamma^{456}t}\eta_0. \qquad \text{(A.10)}$$

In terms of the redefined fields, the dynamical supersymmetry transformation becomes

$$
\begin{aligned}
\delta\hat{X}^i &= i\hat{\Psi}^T\Gamma^i\hat{\epsilon}(t), \\
\delta\hat{A}_t &= Ri\hat{\Psi}^T\hat{\epsilon}(t), \\
\delta\hat{\Psi} &= \left\{\frac{1}{2R}(D_t\hat{X}^i)\Gamma^i + \frac{i}{4}[\hat{X}^i,\hat{X}^j]\Gamma^{ij} + \frac{\mu}{6R}\hat{X}^a\Gamma^a\Gamma^{456}\right. \\
&\quad \left. -\frac{1}{2R}\left(\frac{\mu}{6}+\alpha\right)\left(\hat{X}^2\Gamma^3 - \hat{X}^3\Gamma^2\right)\right\}\hat{\epsilon}(t),
\end{aligned} \tag{A.11}
$$

where $\hat{\epsilon}$ is given by $\epsilon(t) = e^{\frac{1}{2}\Gamma^{23}\alpha t}\hat{\epsilon}(t)$. Here we used $\Gamma^{456} = \Gamma^{23}$ to obtain

$$
\hat{X}^{a'}\Gamma^{a'}\Gamma^{456} = \left(\hat{X}^2\Gamma^2 + \hat{X}^3\Gamma^3\right)\Gamma^{23} = \hat{X}^2\Gamma^3 - \hat{X}^3\Gamma^2. \tag{A.12}
$$

For the case that $\hat{X}^2$ appears only in the adjoint form in the supersymmetry transformation, Taylor's T-duality procedure keeps the supersymmetry. It uniquely fixes the choice of α to

$$
\alpha = -\frac{\mu}{6}. \tag{A.13}
$$

Then, the parameter $\hat{\epsilon}(t)$ becomes t-independent:

$$
\hat{\epsilon}(t) = e^{-\frac{1}{2}\Gamma^{23}\alpha t}\epsilon(t) = e^{\frac{\mu}{12}(\Gamma^{23}-\Gamma^{456})t}\epsilon_0 = \epsilon_0, \tag{A.14}
$$

and the dynamical supersymmetry is expressed as

$$
\begin{aligned}
\delta\hat{X}^i &= i\hat{\Psi}^T\Gamma^i\epsilon_0, \\
\delta\hat{A}_t &= Ri\hat{\Psi}^T\epsilon_0, \\
\delta\hat{\Psi} &= \left\{\frac{1}{2R}(D_t\hat{X}^i)\Gamma^i + \frac{i}{4}[\hat{X}^i,\hat{X}^j]\Gamma^{ij} + \frac{\mu}{6R}\hat{X}^a\Gamma^a\Gamma^{456}\right\}\epsilon_0.
\end{aligned} \tag{A.15}
$$

Also, the kinematical supersymmetry becomes

$$
\tilde{\delta}\hat{\Psi} = \hat{\eta}(t)\mathbf{1}_N \tag{A.16}
$$

with

$$
\hat{\eta}(t) = e^{-\frac{1}{2}\Gamma^{23}\alpha t}\eta(t) = e^{\frac{\mu}{4}\left(\Gamma^{456}+\frac{1}{3}\Gamma^{23}\right)t}\eta_0 = e^{\frac{\mu}{3}\Gamma^{23}t}\eta_0. \tag{A.17}
$$

The final form of the action (A.6) with the surface term dropped is

$$
\begin{aligned}
S &= R\int dt\,\mathrm{Tr}\left[\frac{1}{2R^2}(D_t\hat{X}^i)^2 + \frac{i}{R}\hat{\Psi}^T D_t\hat{\Psi} + \hat{\Psi}^T\Gamma^i[\hat{X}^i,\hat{\Psi}] + \frac{1}{4}[\hat{X}^i,\hat{X}^j]^2\right. \\
&\quad \left. -i\frac{\mu}{3R}\hat{\Psi}^T\Gamma^{23}\hat{\Psi} - \frac{\mu}{3R^2}\hat{X}^3D_t\hat{X}^2 - \frac{1}{2}\left(\frac{\mu}{3R}\right)^2(\hat{X}^a)^2 - i\frac{\mu}{3R}\epsilon_{abc}\hat{X}^a\hat{X}^b\hat{X}^c\right].
\end{aligned} \tag{A.18}
$$

A.2 Uplift to two dimensions: Taylor's T-duality

In order to obtain a two-dimensional theory, we "compactify" the $\hat{X}^2$ direction to a circle of a radius $\hat{R}$ as

$$\hat{X}^2 \sim \hat{X}^2 + 2\pi\hat{R}. \tag{A.19}$$

By using the argument by Taylor [54], the action (A.18) is lifted to two dimensions,

$$\begin{aligned} S \;=\; & R\hat{R}\int dt \int_0^{1/\hat{R}} d\sigma\, \mathrm{Tr}\,\Bigg[\frac{1}{2R^2}F_{t\sigma}^2 + \frac{1}{2R^2}(D_tX^I)^2 - \frac{1}{2}(D_\sigma X^I)^2 \\ & + \frac{i}{R}\Psi^T D_t\Psi - i\Psi^T\Gamma^2 D_\sigma\Psi + \Psi^T\Gamma^I[X^I,\Psi] + \frac{1}{4}[X^I,X^J]^2 \\ & - \frac{1}{2}\left(\frac{\mu}{3R}\right)^2 (X^a)^2 - i\frac{\mu}{3R}\Psi^T\Gamma^{23}\Psi - \frac{\mu}{3R^2}X^3F_{t\sigma} - i\frac{\mu}{3R}\epsilon_{abc}X^aX^bX^c\Bigg], \end{aligned} \tag{A.20}$$

where $I = 3,4,5,6$. (The hats of the fields were omitted.)

By setting

$$\begin{aligned} & t = \frac{1}{R}x_1, \quad A_t = RA_1, \quad \sigma = x_2, \\ & 1/\hat{R} = L, \quad g = (2L)^{1/2}, \quad M = \frac{\mu}{R}, \end{aligned} \tag{A.21}$$

and by rescaling the fermion as $\Psi \to \frac{1}{\sqrt{2}}\Psi$, we obtain

$$\begin{aligned} S \;=\; & \frac{2}{g^2}\int d^2x\, \mathrm{Tr}\,\Bigg[\frac{1}{2}F_{12}^2 + \frac{1}{2}(D_1X^I)^2 - \frac{1}{2}(D_2X^I)^2 \\ & + \frac{i}{2}\Psi^T\left(D_1 - \Gamma^2 D_2\right)\Psi + \frac{1}{2}\Psi^T\Gamma^I[X^I,\Psi] + \frac{1}{4}[X^I,X^J]^2 \\ & - \frac{1}{2}\left(\frac{M}{3}\right)^2 (X^a)^2 - i\frac{M}{6}\Psi^T\Gamma^{23}\Psi - \frac{M}{3}X^3F_{12} - i\frac{M}{3}\epsilon_{abc}X^aX^bX^c\Bigg], \end{aligned} \tag{A.22}$$

where

$$\int d^2x \cdots \equiv \int dx_1 \int_0^L dx_2 \cdots . \tag{A.23}$$

The dynamical supersymmetry is expressed as

$$\begin{aligned} \delta A_1 &= -i\Psi^T\epsilon, \\ \delta A_2 &= -i\Psi^T\Gamma^2\epsilon, \\ \delta X^I &= -i\Psi^T\Gamma^I\epsilon, \end{aligned}$$

$$\delta\Psi = -\Big\{ F_{12}\Gamma^2 + (D_1X^I)\Gamma^I + (D_2X^I)\Gamma^{2I} + \frac{i}{2}[X^I, X^J]\Gamma^{IJ} + \frac{M}{3}X^a\Gamma^a\Gamma^{456}\Big\}\epsilon, \tag{A.24}$$

where $\epsilon = -\epsilon_0/\sqrt{2}$ is an 8-component constant spinor. Note that all the dynamical supersymmetries are preserved. It is in sharp contrast with the case of sixteen supercharges [53], where a half of supersymmetries are broken by deformations.

The kinematical supersymmetry also remains. It is given by

$$\tilde{\delta}\Psi = \hat{\eta}(x_1)\mathbf{1}_N \tag{A.25}$$

with

$$\hat{\eta}(x_1) = e^{\frac{M}{3}\Gamma^{23}x_1}\eta_0, \tag{A.26}$$

where η_0 is constant.

A.2.1 Wick rotation

In order to obtain the Euclidean action, which is going to be put on a lattice, we perform the Wick rotation,

$$x_1 \to -ix_1, \qquad A_1 \to iA_1. \tag{A.27}$$

The Euclidean action is

$$\begin{aligned} S_E = \frac{2}{g^2}\int d^2x\,\mathrm{Tr}\,\Big[&\frac{1}{2}F_{12}^2 + \frac{1}{2}(D_\mu X^I)^2 + \frac{1}{2}\Psi^T(D_1+\gamma_2 D_2)\Psi \\ &+\frac{i}{2}\Psi^T\gamma_I[X^I,\Psi] - \frac{1}{4}[X^I,X^J]^2 \\ &+\frac{1}{2}\left(\frac{M}{3}\right)^2(X^a)^2 - i\frac{M}{6}\Psi^T\gamma_{23}\Psi + i\frac{M}{3}X^3F_{12} + i\frac{M}{3}\epsilon_{abc}X^aX^bX^c\Big], \end{aligned} \tag{A.28}$$

where $\mu = 1, 2$, and $\gamma_I = i\Gamma^I$, which is identical to (3.1).

Then, the dynamical supersymmetry transformation is written as

$$\begin{aligned} \delta A_1 &= \epsilon^T\Psi, \\ \delta A_2 &= \epsilon^T\gamma_2\Psi, \\ \delta X^I &= \epsilon^T\gamma_I\Psi, \\ \delta\Psi &= \left(-F_{12}\gamma_2 - (D_1X^I)\gamma_I + (D_2X^I)\gamma_{2I} + \frac{i}{2}[X^I,X^J]\gamma_{IJ} - \frac{M}{3}X^a\gamma_a\gamma_{456}\right)\epsilon, \end{aligned} \tag{A.29}$$

and the kinematical supersymmetry is given by

$$\delta'\Psi = \hat{\eta}(x_1)\mathbf{1}_N \tag{A.30}$$

with

$$\hat{\eta}(x_1) = e^{i\frac{M}{3}\gamma_{23}x_1}\eta_0. \tag{A.31}$$

B Explicit form of the lattice action

In this section, we explicitly write down the lattice action (4.7) in terms of lattice fields.

We divide the action into the bosonic and the fermionic parts;

$$S_{\rm lat} = S_{\rm lat}^{(B)} + S_{\rm lat}^{(F)}. \tag{B.1}$$

The bosonic part is given by

$$\begin{aligned}
S_{\rm lat}^{(B)} =& \frac{1}{g_0^2}\sum_x \mathrm{Tr}\Bigg[H(x)^2 - 2iH(x)A(x) + \widetilde{H}_\mu(x)^2 + 2i\widetilde{H}_\mu(x)\widetilde{A}_\mu(x) \\
& + \left(\mathcal{D}_\mu\phi_+(x)\right)\left(\mathcal{D}_\mu\phi_-(x)\right) + \frac{1}{4}\left(\mathcal{D}_\mu C(x)\right)^2 \\
& - \frac{1}{4}\left[B(x), C(x)\right]^2 - \left[B(x),\phi_+(x)\right]\left[B(x),\phi_-(x)\right] \\
& + \frac{1}{4}\left[\phi_+(x),\phi_-(x)\right]^2 - \frac{1}{4}\left[C(x),\phi_+(x)\right]\left[C(x),\phi_-(x)\right] \\
& - \frac{M_0}{2}C(x)\left[\phi_+(x),\phi_-(x)\right] + \frac{M_0^2}{9}\left(\frac{1}{4}C(x)^2 + \phi_+(x)\phi_-(x)\right) \\
& - \frac{M_0 m_0}{6}B(x)^2 + i\frac{M_0}{3}B(x)\hat{\Phi}(x)\Bigg],
\end{aligned} \tag{B.2}$$

where

$$\begin{aligned}
A(x) =& \frac{1}{2}\hat{\Phi}(x) + \frac{i}{2}m_0 B(x), \\
\widetilde{A}_1(x) =& \frac{1}{2}\frac{1}{1-\frac{1}{\epsilon^2}||1-U_{12}(x)||^2} \\
& \times\Big[-U_{12}(x)B(x) - B(x)U_{21}(x) \\
& \qquad + U_2(x-\hat{2})^{-1}\left(B(x-\hat{2})U_{12}(x-\hat{2}) + U_{21}(x-\hat{2})B(x-\hat{2})\right)U_2(x-\hat{2})\Big] \\
& + \frac{1}{2\epsilon^2}\frac{\mathrm{Tr}\Big(B(x)\left(U_{12}(x)-U_{21}(x)\right)\Big)}{\left(1-\frac{1}{\epsilon^2}||1-U_{12}(x)||^2\right)^2}\Big(U_{12}(x)-U_{21}(x)\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2\epsilon^2}\frac{\mathrm{Tr}\left(B(x-\hat{2})\left(U_{12}(x-\hat{2})-U_{21}(x-\hat{2})\right)\right)}{\left(1-\frac{1}{\epsilon^2}||1-U_{12}(x-\hat{2})||^2\right)^2}\\
&\qquad\times\left(U_2(x-\hat{2})^{-1}\left(U_{12}(x-\hat{2})-U_{21}(x-\hat{2})\right)U_2(x-\hat{2})\right),\\
\widetilde{A}_2(x)=&\frac{1}{2}\frac{1}{1-\frac{1}{\epsilon^2}||1-U_{12}(x)||^2}\\
&\times\Big[B(x)U_{12}(x)+U_{21}(x)B(x)\\
&\qquad-U_1(x-\hat{1})^{-1}\left(U_{12}(x-\hat{1})B(x-\hat{1})+B(x-\hat{1})U_{21}(x-\hat{1})\right)U_1(x-\hat{1})\Big]\\
&-\frac{1}{2\epsilon^2}\frac{\mathrm{Tr}\left(B(x)\left(U_{12}(x)-U_{21}(x)\right)\right)}{\left(1-\frac{1}{\epsilon^2}||1-U_{12}(x)||^2\right)^2}\Big(U_{12}(x)-U_{21}(x)\Big)\\
&+\frac{1}{2\epsilon^2}\frac{\mathrm{Tr}\left(B(x-\hat{1})\left(U_{12}(x-\hat{1})-U_{21}(x-\hat{1})\right)\right)}{\left(1-\frac{1}{\epsilon^2}||1-U_{12}(x-\hat{1})||^2\right)^2}\\
&\qquad\times\left(U_1(x-\hat{1})^{-1}\left(U_{12}(x-\hat{1})-U_{21}(x-\hat{1})\right)U_1(x-\hat{1})\right).
\end{aligned}
\tag{B.3}
$$

Note that $\widetilde{A}_\mu(x)$ are hermitian. After integrating out the auxiliary fields, $S_{\mathrm{lat}}^{(B)}$ becomes

$$
S_{\mathrm{lat}}^{(B)}=\frac{1}{g_0^2}\sum_x\mathrm{Tr}\left[-\frac{m_0}{2}\left(\frac{M_0}{3}+\frac{m_0}{2}\right)B(x)^2+i\left(\frac{M_0}{3}+\frac{m_0}{2}\right)B(x)\hat{\Phi}(x)\right]+S_{\mathrm{PDT}},\tag{B.4}
$$

where S_{PDT} denotes positive (semi-)definite terms:

$$
\begin{aligned}
S_{\mathrm{PDT}}=&\frac{1}{g_0^2}\sum_x\mathrm{Tr}\Big[\frac{1}{4}\hat{\Phi}(x)^2+\left(\mathcal{D}_\mu\phi_+(x)\right)\left(\mathcal{D}_\mu\phi_-(x)\right)+\frac{1}{4}\left(\mathcal{D}_\mu C(x)\right)^2\\
&-\frac{1}{4}\left[B(x),C(x)\right]^2-\left[B(x),\phi_+(x)\right]\left[B(x),\phi_-(x)\right]\\
&+\frac{1}{4}\left[\phi_+(x),\phi_-(x)\right]^2-\frac{1}{4}\left[C(x),\phi_+(x)\right]\left[C(x),\phi_-(x)\right]\\
&-\frac{M_0}{2}C(x)\left[\phi_+(x),\phi_-(x)\right]+\frac{M_0^2}{9}\left(\frac{1}{4}C(x)^2+\phi_+(x)\phi_-(x)\right)+\widetilde{A}_1(x)^2+\widetilde{A}_2(x)^2\Big].
\end{aligned}
\tag{B.5}
$$

In order that the field $B(x)$ has positive mass squared, m_0 must satisfy

$$
-\frac{2M_0}{3}<m_0<0.\tag{B.6}
$$

The fermionic part is given by

$$
S_{\mathrm{lat}}^{(F)}=\frac{1}{g_0^2}\sum_x\mathrm{Tr}\Big[i\psi_{+\mu}(x)\mathcal{D}_\mu\eta_-(x)+i\psi_{-\mu}(x)\mathcal{D}_\mu\eta_+(x)
$$

$$
\begin{aligned}
&+ \chi_+(x)[\phi_-(x), \chi_+(x)] - \chi_-(x)[\phi_+(x), \chi_-(x)] + \chi_+(x)[C(x), \chi_-(x)] \\
&- \eta_+(x)[B(x), \chi_-(x)] - \eta_-(x)[B(x), \chi_+(x)] \\
&+ \frac{1}{4}\eta_+(x)[\phi_-(x), \eta_+(x)] - \frac{1}{4}\eta_-(x)[\phi_+(x), \eta_-(x)] - \frac{1}{4}\eta_+(x)[C(x), \eta_-(x)] \\
&- \psi_{+\mu}(x)\psi_{+\mu}(x)\Big(\phi_-(x) + U_\mu(x)\phi_-(x+\hat{\mu})U_\mu(x)^{-1}\Big) \\
&+ \psi_{-\mu}(x)\psi_{-\mu}(x)\Big(\phi_+(x) + U_\mu(x)\phi_+(x+\hat{\mu})U_\mu(x)^{-1}\Big) \\
&- \frac{1}{2}\left\{\psi_{+\mu}(x), \psi_{-\mu}(x)\right\}\Big(C(x) + U_\mu(x)C(x+\hat{\mu})U_\mu(x)^{-1}\Big) \\
&+ \frac{1}{2}\psi_{+\mu}(x)\psi_{+\mu}(x)\psi_{-\mu}(x)\psi_{-\mu}(x) \\
&+ \frac{2M_0}{3}\psi_{+\mu}(x)\psi_{-\mu}(x) + \left(\frac{2M_0}{3} + m_0\right)\chi_+(x)\chi_-(x) - \frac{M_0}{6}\eta_+(x)\eta_-(x) \\
&+ i\chi_-(x)\Big(Q_+\hat{\Phi}(x)\Big) - i\chi_+(x)\Big(Q_-\hat{\Phi}(x)\Big) - iB(x)\Big(Q_+Q_-\hat{\Phi}(x)\Big)\Big|_{\text{fermion}}\Bigg],
\end{aligned}
\tag{B.7}
$$

where

$$
\begin{aligned}
Q_\pm\hat{\Phi}(x) =& \frac{-1}{1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2} \\
&\times \Big[-\left(\psi_{\pm 1}(x) + U_1(x)\psi_{\pm 2}(x+\hat{1})U_1(x)^{-1}\right)U_{12}(x) \\
&\quad - U_{21}(x)\left(\psi_{\pm 1}(x) + U_1(x)\psi_{\pm 2}(x+\hat{1})U_1(x)^{-1}\right) \\
&\quad + \left(\psi_{\pm 2}(x) + U_2(x)\psi_{\pm 1}(x+\hat{2})U_2(x)^{-1}\right)U_{21}(x) \\
&\quad + U_{12}(x)\left(\psi_{\pm 2}(x) + U_2(x)\psi_{\pm 1}(x+\hat{2})U_2(x)^{-1}\right)\Big] \\
&+ \frac{U_{12}(x) - U_{21}(x)}{\left(1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2\right)^2}\frac{1}{\epsilon^2} \\
&\times \mathrm{Tr}\left[\Big(U_{12}(x) - U_{21}(x)\Big)\Big(\mathcal{D}_2\psi_{\pm 1}(x) - \mathcal{D}_1\psi_{\pm 2}(x)\Big)\right],
\end{aligned}
\tag{B.8}
$$

$$
\begin{aligned}
Q_+Q_-\hat{\Phi}(x)\Big|_{\text{fermion}} =& -i\frac{Q_+Q_-\left(U_{12}(x) - U_{21}(x)\right)\Big|_{\text{fermion}}}{1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2} \\
&+ \frac{i\left(U_{12}(x) - U_{21}(x)\right)}{\Big(1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2\Big)^2}\frac{1}{\epsilon^2}\mathrm{Tr}\Big[Q_+Q_-\left(U_{12}(x) + U_{21}(x)\right)\Big]\Big|_{\text{fermion}} \\
&- \frac{1}{\Big(1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2\Big)^2}\frac{1}{\epsilon^2} \\
&\times \Big\{-iQ_+\left(U_{12}(x) - U_{21}(x)\right)\mathrm{Tr}\Big[Q_-\left(U_{12}(x) + U_{21}(x)\right)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\qquad + iQ_-\left(U_{12}(x) - U_{21}(x)\right) \mathrm{Tr}\left[Q_+\left(U_{12}(x) + U_{21}(x)\right)\right]\Bigg\} \\
&- 2\frac{i\left(U_{12}(x) - U_{21}(x)\right)}{\left(1 - \frac{1}{\epsilon^2}||1 - U_{12}(x)||^2\right)^3}\frac{1}{\epsilon^4} \\
&\times \mathrm{Tr}\left[Q_+\left(U_{12}(x) + U_{21}(x)\right)\right] \mathrm{Tr}\left[Q_-\left(U_{12}(x) + U_{21}(x)\right)\right],
\end{aligned}
\tag{B.9}
$$

with

$$
\begin{aligned}
&Q_+Q_-\left(U_{12}(x) - U_{21}(x)\right)\Big|_{\mathrm{fermion}} \\
&= \Big\{-\frac{1}{2}\left[\psi_{+1}(x), \psi_{-1}(x)\right] - \frac{1}{2}U_1(x)[\psi_{+2}(x+\hat{1}), \psi_{-2}(x+\hat{1})]U_1(x)^{-1} \\
&\qquad - \psi_{+1}(x)U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1} + \psi_{-1}(x)U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\Big\}U_{12}(x) \\
&+ U_{12}(x)\Big\{-\frac{1}{2}\left[\psi_{+2}(x), \psi_{-2}(x)\right] - \frac{1}{2}U_2(x)[\psi_{+1}(x+\hat{2}), \psi_{-1}(x+\hat{2})]U_2(x)^{-1} \\
&\qquad - U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\psi_{-2}(x) + U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1}\psi_{+2}(x)\Big\} \\
&- \Big\{-\frac{1}{2}\left[\psi_{+2}(x), \psi_{-2}(x)\right] - \frac{1}{2}U_2(x)[\psi_{+1}(x+\hat{2}), \psi_{-1}(x+\hat{2})]U_2(x)^{-1} \\
&\qquad - \psi_{+2}(x)U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1} + \psi_{-2}(x)U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\Big\}U_{21}(x) \\
&- U_{21}(x)\Big\{-\frac{1}{2}\left[\psi_{+1}(x), \psi_{-1}(x)\right] - \frac{1}{2}U_1(x)[\psi_{+2}(x+\hat{1}), \psi_{-2}(x+\hat{1})]U_1(x)^{-1} \\
&\qquad - U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\psi_{-1}(x) + U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1}\psi_{+1}(x)\Big\} \\
&- \left(\psi_{-1}(x) + U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1}\right)U_{12}(x) \\
&\qquad\qquad \times \left(\psi_{+2}(x) + U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\right) \\
&+ \left(\psi_{+1}(x) + U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\right)U_{12}(x) \\
&\qquad\qquad \times \left(\psi_{-2}(x) + U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1}\right) \\
&+ \left(\psi_{-2}(x) + U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1}\right)U_{21}(x) \\
&\qquad\qquad \times \left(\psi_{+1}(x) + U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\right) \\
&- \left(\psi_{+2}(x) + U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\right)U_{21}(x) \\
&\qquad\qquad \times \left(\psi_{-1}(x) + U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1}\right),
\end{aligned}
\tag{B.10}
$$

$$
\begin{aligned}
&\mathrm{Tr}\left[Q_+Q_-\left(U_{12}(x) + U_{21}(x)\right)\right]\Big|_{\mathrm{fermion}} \\
&= \mathrm{Tr}\Bigg[U_{12}(x)\Big\{-\left\{\psi_{+1}(x), \psi_{-2}(x)\right\} + \left\{\psi_{-1}(x), \psi_{+2}(x)\right\} \\
&\qquad + \left(\mathcal{D}_2\psi_{+1}(x)\right)\left(\mathcal{D}_1\psi_{-2}(x)\right) - \left(\mathcal{D}_2\psi_{-1}(x)\right)\left(\mathcal{D}_1\psi_{+2}(x)\right) \\
&\qquad + \frac{1}{2}\left(\mathcal{D}_2\psi_{+1}(x)\right)\psi_{-1}(x) + \frac{1}{2}U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1}\left(\mathcal{D}_2\psi_{+1}(x)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad -\frac{1}{2}\left(\mathcal{D}_2\psi_{-1}(x)\right)\psi_{+1}(x)-\frac{1}{2}U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\left(\mathcal{D}_2\psi_{-1}(x)\right)\\
&\quad +\frac{1}{2}\psi_{+2}(x)\left(\mathcal{D}_1\psi_{-2}(x)\right)+\frac{1}{2}\left(\mathcal{D}_1\psi_{-2}(x)\right)U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\\
&\quad \left.-\frac{1}{2}\psi_{-2}(x)\left(\mathcal{D}_1\psi_{+2}(x)\right)-\frac{1}{2}\left(\mathcal{D}_1\psi_{+2}(x)\right)U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1}\right\}\\
&+U_{21}(x)\Big\{\{\psi_{+1}(x),\psi_{-2}(x)\}-\{\psi_{-1}(x),\psi_{+2}(x)\}\\
&\quad +\left(\mathcal{D}_1\psi_{+2}(x)\right)\left(\mathcal{D}_2\psi_{-1}(x)\right)-\left(\mathcal{D}_1\psi_{-2}(x)\right)\left(\mathcal{D}_2\psi_{+1}(x)\right)\\
&\quad +\frac{1}{2}\psi_{+1}(x)\left(\mathcal{D}_2\psi_{-1}(x)\right)+\frac{1}{2}\left(\mathcal{D}_2\psi_{-1}(x)\right)U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\\
&\quad -\frac{1}{2}\psi_{-1}(x)\left(\mathcal{D}_2\psi_{+1}(x)\right)-\frac{1}{2}\left(\mathcal{D}_2\psi_{+1}(x)\right)U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1}\\
&\quad +\frac{1}{2}\left(\mathcal{D}_1\psi_{+2}(x)\right)\psi_{-2}(x)+\frac{1}{2}U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1}\left(\mathcal{D}_1\psi_{+2}(x)\right)\\
&\quad \left.\left.-\frac{1}{2}\left(\mathcal{D}_1\psi_{-2}(x)\right)\psi_{+2}(x)-\frac{1}{2}U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\left(\mathcal{D}_1\psi_{-2}(x)\right)\right\}\right],
\end{aligned}
\tag{B.11}
$$

$$
\begin{aligned}
&-iQ_+\left(U_{12}(x)-U_{21}(x)\right)\mathrm{Tr}\left[Q_-\left(U_{12}(x)+U_{21}(x)\right)\right]\\
&\quad +iQ_-\left(U_{12}(x)-U_{21}(x)\right)\mathrm{Tr}\left[Q_+\left(U_{12}(x)+U_{21}(x)\right)\right]\\
&\quad =\Big\{\big(\psi_{+1}(x)+U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\big)U_{12}(x)\\
&\qquad -U_{12}(x)\big(\psi_{+2}(x)+U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\big)\\
&\qquad -\big(\psi_{+2}(x)+U_2(x)\psi_{+1}(x+\hat{2})U_2(x)^{-1}\big)U_{21}(x)\\
&\qquad +U_{21}(x)\big(\psi_{+1}(x)+U_1(x)\psi_{+2}(x+\hat{1})U_1(x)^{-1}\big)\Big\}\\
&\qquad \times\mathrm{Tr}\left[i\left(U_{12}(x)-U_{21}(x)\right)\Big(-\mathcal{D}_2\psi_{-1}(x)+\mathcal{D}_1\psi_{-2}(x)\Big)\right]\\
&\quad -\Big\{\big(\psi_{-1}(x)+U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1}\big)U_{12}(x)\\
&\qquad -U_{12}(x)\big(\psi_{-2}(x)+U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1}\big)\\
&\qquad -\big(\psi_{-2}(x)+U_2(x)\psi_{-1}(x+\hat{2})U_2(x)^{-1}\big)U_{21}(x)\\
&\qquad +U_{21}(x)\big(\psi_{-1}(x)+U_1(x)\psi_{-2}(x+\hat{1})U_1(x)^{-1}\big)\Big\}\\
&\qquad \times\mathrm{Tr}\left[i\left(U_{12}(x)-U_{21}(x)\right)\Big(-\mathcal{D}_2\psi_{+1}(x)+\mathcal{D}_1\psi_{+2}(x)\Big)\right],
\end{aligned}
\tag{B.12}
$$

$$
\mathrm{Tr}\left[Q_+\left(U_{12}(x)+U_{21}(x)\right)\right]\mathrm{Tr}\left[Q_-\left(U_{12}(x)+U_{21}(x)\right)\right]
$$

$$
\begin{aligned}
&= \mathrm{Tr}\left[i\left(U_{12}(x) - U_{21}(x)\right)\Big(-\mathcal{D}_2\psi_{+1}(x) + \mathcal{D}_1\psi_{+2}(x)\Big)\right] \\
&\times \mathrm{Tr}\left[i\left(U_{12}(x) - U_{21}(x)\right)\Big(-\mathcal{D}_2\psi_{-1}(x) + \mathcal{D}_1\psi_{-2}(x)\Big)\right].
\end{aligned} \tag{B.13}
$$

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